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Weighted finite difference methods for a class of space fractional partial differential equations with variable coefficients[☆]

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ABSTRACT

A class of weighted finite difference methods (WFDMs) for solving a class of initial-boundary value problems of space fractional partial differential equations with variable coefficients is presented. Their stability and convergence properties are considered. It is proven that the WFDMs are unconditionally-stable for $0 \le r \le \frac{1}{2}$, and conditionally-stable for $\frac{1}{2} < r \le 1$, here r is the weighting parameter subjected to $0 \le r \le 1$. Some convergence results are given. These methods, problems and results generalize the corresponding methods, problems and results given in [7,8,10]. Some numerical examples are provided to show the effectiveness of the methods with different weighting parameters.

1. Introduction

Fractional derivatives provide a powerful tool for the description of memory and hereditary properties of different substances because of their non-locality property. In recent years, fractional partial differential equations (FPDEs) have played a key role in modeling particle transport in anomalous diffusion in many diverse fields, including finance, semiconductor, biology, hydrogeology, physics, electrical engineering and control theory (cf. [1–5]). Space fractional partial differential equations (SFPDEs) are used to model super-diffusion, where a particle plume spread faster than the classical Brownian motion model predicts. Some different numerical methods for solving SFPDEs have been proposed. Liu, Anh and Turner [6] firstly proposed Method of Lines (MOL) to transform the space fractional Fokker–Planck equation into a system of ordinary differential equations. Meerschaert et al. [7–9] and Tadjeran et al. [10,11] proposed three kinds of finite difference approximations which are the implicit Euler method, the explicit Euler method and the fractional Cranck–Nicholson method for partial SFPDEs based on shifted Grünwald formula and derived some detailed stability and convergence analysis. Fix and Roop [12] developed a least squares finite element solution of a fractional-order two-point boundary value problem. Lynch et al. [13] constructed two different numerical methods, but stability and convergence analysis were not given. Shen

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and Liu [14] gave an explicit difference approximation for space fractional diffusion equation. Liu et al. [15] discussed an approximation of the Lévy–Feller advection–dispersion process by random walk and finite difference method.

In this paper, we consider the numerical methods for a class of SFPDEs with variable coefficients:

$$\frac{\partial u(x,t)}{\partial t} = -v(x)\frac{\partial u(x,t)}{\partial x} + d_+(x)\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} + d_-(x)\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} + f(x,t), \tag{1.1}$$

$$u(x, t = 0) = s(x), \tag{1.2}$$

$$u(L,t) = 0, \qquad u(R,t) = b(t), \tag{1.3}$$

where $L < x < R$, $0 \le t \le T$, $1 < \alpha \le 2$, $v(x) \ge 0$, $d_-(x) \ge 0$, $d_+(x) \ge 0$, $f(x,t)$ is a source term.

The left-handed($+$) and the right-handed($-$) fractional derivatives in (1.1) are the Riemann–Liouville partial fractional derivatives of order α defined by (cf. [8,4])

$$\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} = \frac{1}{\Gamma(\tilde{n}-\alpha)}\frac{\partial^{\tilde{n}}}{\partial x^{\tilde{n}}}\int_L^x u(\xi,t)\frac{\mathrm{d}\xi}{(x-\xi)^{\alpha+1-\tilde{n}}},$$

$$\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} = \frac{(-1)^{\tilde{n}}}{\Gamma(\tilde{n}-\alpha)}\frac{\partial^{\tilde{n}}}{\partial x^{\tilde{n}}}\int_x^R u(\xi,t)\frac{\mathrm{d}\xi}{(\xi-x)^{\alpha+1-\tilde{n}}},$$

where $\tilde{n}$ is an integer such that $\tilde{n} - 1 < \alpha \le \tilde{n}$.

For SFPDEs with variable coefficients, analytical solutions are hard to be derived by using Laplace–Fourier transform methods, so we have to resort to numerical solutions. Moreover, the fractional-order models with variable coefficients have been used in some practical problems (cf. [16]). The Eq. (1.1) include the fractional-order advection–dispersion transport equations solved in [7], the space fractional diffusion equation solved in [13,10], the two-sided space fractional differential equation solved in [8] and the special space fractional Fokker–Planck equation Example 5.2 given in [6].

If $\alpha = m$, then the above definitions turn out to be the standard integer derivatives:

$$\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} = \frac{\partial^m u(x,t)}{\partial x^m}, \qquad \frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} = (-1)^m\frac{\partial^m u(x,t)}{\partial x^m}.$$

Let $\alpha = 2$, $d(x) = d_+(x) + d_-(x)$, Eq. (1.1) becomes the following convection–diffusion equation with a source term

$$\frac{\partial u(x)}{\partial t} = -v(x)\frac{\partial u(x,t)}{\partial x} + d(x)\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t). \tag{1.4}$$

If $d_-(x) = 0$, then Eq. (1.1) has no right-handed fractional derivative, and the left-handed fractional derivative is usually noted as $\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}$. Therefore, Eq. (1.1) becomes space fractional advection–dispersion equation (cf. [7])

$$\frac{\partial u(x,t)}{\partial t} = -v(x)\frac{\partial u(x,t)}{\partial x} + d_+(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + f(x,t). \tag{1.5}$$

If $v(x) = 0$, $d_-(x) = 0$, Eq. (1.1) becomes space fractional diffusion equation (cf. [13,10])

$$\frac{\partial u(x,t)}{\partial t} = d_+(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + f(x,t). \tag{1.6}$$

If $v(x) = 0$, Eq. (1.1) reduces to two-sided space fractional differential equation (cf. [8])

$$\frac{\partial u(x,t)}{\partial t} = d_+(x)\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} + d_-(x)\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} + f(x,t). \tag{1.7}$$

Eq. (1.7) with the case $1 < \alpha < 2$ can be used to model a super-diffusive process, where particles diffuse faster than the classical model (1.4) predicts.

Meerschaert and Tadjeran [7] proposed the implicit Euler method for Eq. (1.5), and they studied Eq. (1.7) (we only consider coefficients that depend on a space variable here) and derived the explicit and implicit Euler methods (cf. [8]). Tadjeran et al. [10] gave the fractional Cranck–Nicholson method for Eq. (1.6). Yuste [17] extended the weighted average finite difference methods (WAFDMs) for a class of the diffusion equations of time fractional derivative and considered their accuracy and stability.

In this paper, we give a class of WFDMs for Eqs. (1.1)–(1.3), which are the extension of the explicit Euler method, the implicit Euler method and the fractional Cranck–Nicholson method proposed by Meerschaert et al. [7,8] and Tadjeran et al. [10]. There exist some important differences between our WFDMs and Yuste's WAFDMs, which will be compared in Remark 4.2.

2. A class of WFDMs for SFPDEs

Let τ be the time step, $t_n = n\tau, n = 0, 1, 2, \ldots, 0 \le t_n \le T$, and h be the grid step in space, $x_i = L + ih, L \le x_i \le R$, $i = 0, 1, \ldots, K$. Let u_i^n be the numerical estimate of the value of the exact solution $u(x, t)$ at the mesh point (x_i, t_n). Similarly, we denote $d_{-i} = d_-(x_i), d_{+i} = d_+(x_i), f_i^n = f(x_i, t_n)$.

We apply the shifted Grünwald formula (cf. [7,8]) to discretize the left-handed fractional derivative and the right-handed fractional derivative.

$$\frac{\partial^\alpha u(x_i, t_n)}{\partial_+ x^\alpha} = \frac{1}{h^\alpha}\sum_{k=0}^{i+1} g_k u(x_i - (k-1)h, t_n) + O(h), \tag{2.1}$$

$$\frac{\partial^\alpha u(x_i, t_n)}{\partial_- x^\alpha} = \frac{1}{h^\alpha}\sum_{k=0}^{K-i+1} g_k u(x_i + (k-1)h, t_n) + O(h), \tag{2.2}$$

where the Grünwald coefficients are defined by

$$g_0 = 1, \qquad g_k = (-1)^k \frac{(\alpha)(\alpha-1)\cdots(\alpha-k+1)}{k!} \quad (k = 1, 2, 3, \ldots). \tag{2.3}$$

The first-order time derivative is discretized by using the explicit Euler formula. The first-order space derivative is replaced by a weighted average of the backward difference formula at the time points t_n and t_{n+1}. The left-handed fractional derivative is replaced by a weighted average of the shifted Grünwald formula (2.1) evaluated at the time points t_n and t_{n+1}, and the right-handed fractional derivative is estimated by using a weighted average of the shifted Grünwald formula (2.2) evaluated at the same time points.

The resulting WFDMs take the following form

$$\begin{aligned}\frac{u_i^{n+1} - u_i^n}{\tau} &= -v_i\left(r\frac{u_i^n - u_{i-1}^n}{h} + (1-r)\frac{u_i^{n+1} - u_{i-1}^{n+1}}{h}\right) + \frac{d_{+i}}{h^\alpha}\left[r\sum_{k=0}^{i+1} g_k u_{i-k+1}^n + (1-r)\sum_{k=0}^{i+1} g_k u_{i-k+1}^{n+1}\right] \\ &\quad + \frac{d_{-i}}{h^\alpha}\left[r\sum_{k=0}^{K-i+1} g_k u_{i+k-1}^n + (1-r)\sum_{k=0}^{K-i+1} g_k u_{i+k-1}^{n+1}\right] + rf_i^n + (1-r)f_i^{n+1}\end{aligned}$$

for $i = 0, 1, \ldots, K, n = 1, 2, \ldots$, where r is the weighting parameter subjected to $0 \le r \le 1$. Through arrangement, we obtain

$$\begin{aligned}&u_i^{n+1} - \xi_i(1-r)\sum_{k=0}^{i+1} g_k u_{i-k+1}^{n+1} - \eta_i(1-r)\sum_{k=0}^{K-i+1} g_k u_{i+k-1}^{n+1} + \zeta_i(1-r)(u_i^{n+1} - u_{i-1}^{n+1}) \\ &\quad = u_i^n + \xi_i r\sum_{k=0}^{i+1} g_k u_{i-k+1}^n + \eta_i r\sum_{k=0}^{K-i+1} g_k u_{i+k-1}^n - r\zeta_i(u_i^n - u_{i-1}^n) + [rf_i^n + (1-r)f_i^{n+1}]\tau\end{aligned} \tag{2.4}$$

for $i = 1, 2, \ldots, K-1, n = 1, 2, \ldots$, where

$$\xi_i = \frac{d_{+i}\tau}{h^\alpha} \ge 0, \qquad \eta_i = \frac{d_{-i}\tau}{h^\alpha} \ge 0, \qquad \zeta_i = \frac{v_i\tau}{h} \ge 0.$$

The above weighted finite difference equation (2.4) together with the Dirichlet boundary conditions can be written as a matrix form

$$\begin{aligned}&[I - (1-r)A]U^{n+1} = (I + rA)U^n + Q^n\tau, \\ &U^n = [u_1^n, u_2^n, \ldots, u_{K-1}^n]^{\mathrm{T}},\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}&Q^n\tau = \left[q_1, q_2, \ldots, q_{K-2}, q_{K-1} + \tilde{q}\right], \\ &q_i = [rf_i^n + (1-r)f_i^{n+1}]\tau + \eta_i g_{K-i+1}[(1-r)b(t_{n+1}) + rb(t_n)], \quad i = 1, 2, \ldots, K-1, \\ &\tilde{q} = \xi_{K-1}[(1-r)b(t_{n+1}) + rb(t_n)], \\ &A = (A_{ij}), \quad i = 1, 2, \ldots, K-1, j = 1, 2, \ldots, K-1, \\ &A_{ij} = \begin{cases}(\xi_i + \eta_i)g_1 - \zeta_i, & j = i, \\ \xi_i g_2 + \eta_i g_0 + \zeta_i, & j = i-1, \\ \xi_i g_0 + \eta_i g_2, & j = i+1, \\ \xi_i g_{i-j+1}, & j < i-1, \\ \eta_i g_{j-i+1}, & j > i+1.\end{cases}\end{aligned}$$

Remark 2.1. If $r = 0, d_-(x) = 0$, the WFDMs (2.4) reduce to the implicit Euler method for Eq. (1.5) given in [7]. If $r = \frac{1}{2}, v(x) = d_-(x) = 0$, the WFDMs (2.4) reduce to the fractional Cranck–Nicholson method for Eq. (1.6) given in [10]. If

$v(x) = 0$, the WFDMs (2.4) reduce to the implicit Euler method when $r = 0$ and the explicit Euler method when $r = 1$ for Eq. (1.7) proposed in [8].

3. Stability analysis of WFDMs

In order to perform stability analysis of the WFDMs (2.4), we first give the properties of the Grünwald coefficients (2.3) (cf. [7]).

$$g_1 = -\alpha, \quad g_j \geq 0\,(j \neq 1, 1 \leq \alpha \leq 2), \qquad \sum_{j=0}^{\infty} g_i = 0, \qquad \sum_{k=0,k\neq 1}^{N} g_k \leq -g_1 \quad (N = 1, 2, \ldots). \tag{3.1}$$

Theorem 3.1. (1) *If $0 \leq r \leq \frac{1}{2}$, the methods (2.4) for Eqs. (1.1)–(1.3) are unconditionally-stable.*
(2) *If $\frac{1}{2} < r \leq 1$, the methods (2.4) for Eqs. (1.1)–(1.3) are conditionally-stable when $(\xi + \eta)\alpha + \zeta \leq \frac{1}{2r-1}$, where $\xi = d_{+\max}\tau/h^\alpha$, $\eta = d_{-\max}\tau/h^\alpha$, $\zeta = v_{\max}\tau/h$, and $d_{+\max}, d_{-\max}, v_{\max}$ are the maximum values of $d_+(x), d_-(x), v(x)$ over the region $L \leq x \leq R$.*

Proof. Suppose c^n and u^n are the two solutions of the difference equation (2.5), and the error vector $\varepsilon^n = c^n - u^n$. We only consider the perturbation for the initial condition here. Let

$$[I - (1-r)A]^{-1}(I + rA) := B.$$

From (2.5), we have $\varepsilon^{n+1} = B\varepsilon^n$, moreover,

$$\varepsilon^{n+1} = B^{n+1}\varepsilon^0, \qquad \|\varepsilon^n\| \leq \|B^n\|\|\varepsilon^0\|.$$

According to (3.1), we have

$$A_{ii} = (\xi_i + \eta_i)g_1 - \zeta_i = -(\xi_i + \eta_i)\alpha - \zeta_i,$$

the radius

$$R_i := \sum_{k=1,k\neq i}^{K-1} A_{ik} = \sum_{k=0,k\neq 1}^{i} \xi_i g_k + \sum_{k=0,k\neq 1}^{K-i} \eta_i g_k + \zeta_i \leq (\xi_i + \eta_i)\alpha + \zeta_i.$$

Let λ be an eigenvalue of the matrix A. By using Gerschgorin theorem, for every eigenvalue λ, there exist A_{ii} such that

$$|\lambda - A_{ii}| \leq R_i, \quad i.e.\ |\lambda + (\xi_i + \eta_i)\alpha + \zeta_i| \leq (\xi_i + \eta_i)\alpha + \zeta_i. \tag{3.2}$$

From (3.2), the real parts of the eigenvalues of the matrix A are non-positive. The eigenvalue of the matrix B is $\frac{1+r\lambda}{1-(1-r)\lambda}(0 \leq r \leq 1)$.

If $0 \leq r \leq \frac{1}{2}$, the equality

$$\left|\frac{1 + r\lambda}{1 - (1-r)\lambda}\right| \leq 1 \tag{3.3}$$

holds for any r.

If $\frac{1}{2} < r \leq 1$, the inequality (3.3) is established under the condition

$$(\xi_i + \eta_i)\alpha + \zeta_i \leq \frac{1}{2r-1}. \tag{3.4}$$

It follows from (3.3) that the spectral radius of the matrix B is no larger than 1, then for some positive constant M, $\|B^n\| \leq (\|B\|)^n \leq M < \infty$, and therefore $\|\varepsilon^n\| \leq M\|\varepsilon^0\|$.

Hence, the methods (2.4) are unconditionally-stable for $0 \leq r \leq \frac{1}{2}$, and stable for $\frac{1}{2} < r \leq 1$ under the condition (3.4). □

Remark 3.1. If $v(x) = 0$, $d_-(x) = 0$, then Eq. (1.1) becomes Eq. (1.6). Choosing $r = 1$ in (2.4), the stability condition (3.4) reduces to

$$\frac{\tau}{h^\alpha} \leq \frac{1}{\alpha d_{+\max}};$$

If $v(x) = 0$, Eq. (1.1) becomes Eq. (1.7). When $r = 1$, the stability condition (3.4) becomes

$$\frac{\tau}{h^\alpha} \leq \frac{1}{\alpha(d_{+\max} + d_{-\max})}.$$

The above two results have been given in [8].

4. Convergence analysis

Theorem 4.1. *Let $u(x_j, t_n)$ be the exact solution of Eqs. (1.1)–(1.3) and u_j^n be the solution of the weighted finite difference Eq. (2.4) at the mesh point (x_j, t_n). For the weighting parameter $0 \le r \le 1$, there exists the positive constant $\tilde{C}$ such that*

$$|u_j^n - u(x_j, t_n)| \le \begin{cases} \tilde{C}(\tau + h), & 0 \le r < \dfrac{1}{2}, \\ \tilde{C}(\tau^2 + h), & r = \dfrac{1}{2}, \\ \tilde{C}(\tau + h), & \dfrac{1}{2} < r \le 1, (\xi + \eta)\alpha + \zeta \le \dfrac{1}{2r - 1} \end{cases}$$

for $j = 1, 2, \ldots, K-1$, $n = 1, 2, \ldots$, where ξ, η and ζ are defined in Theorem 3.1.

Proof. We first compute the order of the local truncation error $R_j^n(u)$. In fact,

$$\begin{aligned} R_j^n(u) = {} & \frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{\tau} + v(x_j)\left(r\frac{u(x_j, t_n) - u(x_{j-1}, t_n)}{h} + (1-r)\frac{u(x_j, t_{n+1}) - u(x_{j-1}, t_{n+1})}{h}\right) \\ & - d_+(x_j)\left[rh^{-\alpha}\sum_{k=0}^{j+1} g_k u(x_{j-k+1}, t_n) + (1-r)h^{-\alpha}\sum_{k=0}^{j+1} g_k u(x_{j-k+1}, t_{n+1})\right] \\ & - d_-(x_j)\left[rh^{-\alpha}\sum_{k=0}^{K-j+1} g_k u(x_{j+k-1}, t_n) + (1-r)h^{-\alpha}\sum_{k=0}^{K-j+1} g_k u(x_{j+k-1}, t_{n+1})\right] - rf(x_j, t_n) - (1-r)f(x_j, t_{n+1}), \end{aligned}$$

and

$$\frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{\tau} = \frac{\partial u(x_j, t_n)}{\partial t} + \frac{\tau}{2}\frac{\partial^2 u(x_j, t_n)}{\partial t^2} + O(\tau^2), \tag{4.1}$$

$$\frac{u(x_j, t_n) - u(x_{j-1}, t_n)}{h} = \frac{\partial u(x_j, t_n)}{\partial x} + O(h). \tag{4.2}$$

Using (2.1), (2.2), (4.1) and (4.2), we obtain

$$\begin{aligned} R_j^n(u) = {} & \frac{\partial u(x_j, t_n)}{\partial t} + \frac{\tau}{2}\frac{\partial^2 u(x_j, t_n)}{\partial t^2} + O(\tau^2) \\ & + v(x_j)\left[r\left(\frac{\partial u(x_j, t_n)}{\partial x} + O(h)\right) + (1-r)\left(\frac{\partial u(x_j, t_{n+1})}{\partial x} + O(h)\right)\right] \\ & - d_+(x_j)\left[r\left(\frac{\partial^\alpha u(x_j, t_n)}{\partial_+ x^\alpha} + O(h)\right) + (1-r)\left(\frac{\partial^\alpha u(x_j, t_{n+1})}{\partial_+ x^\alpha} + O(h)\right)\right] \\ & - d_-(x_j)\left[r\left(\frac{\partial^\alpha u(x_j, t_n)}{\partial_- x^\alpha} + O(h)\right) + (1-r)\left(\frac{\partial^\alpha u(x_j, t_{n+1})}{\partial_- x^\alpha} + O(h)\right)\right] - rf(x_j, t_n) - (1-r)f(x_j, t_{n+1}), \\ = {} & \frac{\partial u(x_j, t_n)}{\partial t} + \frac{\tau}{2}\frac{\partial^2 u(x_j, t_n)}{\partial t^2} - r\frac{\partial u(x_j, t_n)}{\partial t} - (1-r)\frac{\partial u(x_j, t_{n+1})}{\partial t} + O(\tau^2 + h) \\ = {} & \left(r - \frac{1}{2}\right)\tau\frac{\partial^2 u(x_j, t_n)}{\partial t^2} + O(\tau^2 + h). \end{aligned}$$

Therefore

$$R_j^n(u) = \begin{cases} O(\tau^2 + h), & r = \dfrac{1}{2}, \\ O(\tau + h), & r \ne \dfrac{1}{2}, \quad 0 \le r \le 1. \end{cases}$$

Next, we compute the order of convergence.
Define $e_j^n = u(x_j, t_n) - u_j^n, j = 1, 2, \ldots, K-1, n = 1, 2, \ldots$. Substitution into (2.4) leads to

$$\begin{aligned} & e_j^{n+1} - \xi_j(1-r)\sum_{k=0}^{j+1} g_k e_{j-k+1}^{n+1} - \eta_j(1-r)\sum_{k=0}^{K-j+1} g_k e_{j+k-1}^{n+1} + \zeta_j(1-r)(e_j^{n+1} - e_{j-1}^{n+1}) \\ & = e_j^n + \xi_j r\sum_{k=0}^{j+1} g_k e_{j-k+1}^{n+1} + \eta_j r\sum_{k=0}^{K-j+1} g_k e_{j+k-1}^{n} - r\zeta_j(e_j^n - e_{j-1}^n) + \tau R_j^n. \end{aligned} \tag{4.3}$$

Considering $e_K^n = 0$, $e_0^n = 0$, $e_j^0 = 0$ ($j = 0, 1, \ldots, K$). Eq. (4.3) can be written in a matrix form

$$E^{n+1} = BE^n + CW, \tag{4.4}$$

where

$$E^n = [e_1^n, e_2^n, \ldots, e_{K-1}^n]^{\mathrm{T}},$$
$$B = [I - (1-r)A]^{-1}(I + rA), \qquad C = [I - (1-r)A]^{-1}.$$

When $r \neq \frac{1}{2}$ $(0 \le r \le 1)$,

$$W = [\tau O(\tau + h), \tau O(\tau + h), \ldots, \tau O(\tau + h)]^{\mathrm{T}}.$$

When $r = \frac{1}{2}$,

$$W = [\tau O(\tau^2 + h), \tau O(\tau^2 + h), \ldots, \tau O(\tau^2 + h)]^{\mathrm{T}}.$$

Recursion from (4.4) yields to

$$E^{n+1} = (B^n + B^{n-1} + \cdots + B + I)CW. \tag{4.5}$$

The following proof is divided into two cases: $r \neq \frac{1}{2}$ $(0 \le r \le 1)$ and $r = \frac{1}{2}$.

If $r \neq \frac{1}{2}$ $(0 \le r \le 1)$, then according to Theorem 3.1, when $0 \le r < \frac{1}{2}$ or $\frac{1}{2} < r \le 1$ and $(\xi + \eta)\alpha + \zeta \le \frac{1}{2r-1}$, we have $\rho(B) \le 1$, and

$$\rho(B^n) \le (\rho(B))^n \le 1. \tag{4.6}$$

It follows from (3.2) that the eigenvalues of the matrix A have non-positive real parts, therefore,

$$\rho(C) \le 1. \tag{4.7}$$

For any given ε,

$$\|B^n\| \le \rho(B^n) + \varepsilon \le 1 + \varepsilon \quad (k = 0, 1, \ldots, n), \tag{4.8}$$
$$\|C\| \le \rho(C) + \varepsilon \le 1 + \varepsilon. \tag{4.9}$$

From (4.8)–(4.9), we have

$$\begin{aligned}\|E^{n+1}\| &\le (\|B^n\| + \|B^{n-1}\| + \cdots + \|B\| + \|I\|)\|C\|\,\|W\| \\ &\le (n+1)(1+\varepsilon)^2 \tau O(\tau + h).\end{aligned}$$

Because $(n+1)\tau \le T$, we obtain

$$\|E^{n+1}\| \le T(1+\varepsilon)^2 O(\tau + h).$$

Therefore, for the case of $r \neq \frac{1}{2}$ $(0 \le r \le 1)$, the methods (2.4) are unconditionally-convergent when $0 \le r < \frac{1}{2}$, and convergent under the condition $(\xi + \eta)\alpha + \zeta \le \frac{1}{2r-1}$ when $\frac{1}{2} < r \le 1$.

For the case of $r = \frac{1}{2}$, the convergence can be proved similarly. We omit it here. □

Remark 4.1. By Theorem 4.1, when $r = \frac{1}{2}$, the WFDMs (2.4) are first-order accurate in space and second-order accurate in time and become the fractional Cranck–Nicholson scheme, and Tadjeran et al. [10] improved its convergence order in space by the spatial extrapolation method and obtained a second-order accurate finite difference method for the fractional diffusion equation. Moreover, when $r \neq \frac{1}{2}$, the WFDMs (2.4) are first-order accurate in space and time, and can also achieve space second-order accuracy by using the spatial extrapolation method given in [10].

Remark 4.2. Though the WFDMs in this paper and the WAFDMs in [17] all are an extension of the weighted average methods for (non-fractional) ordinary and partial differential equations, there exist some important differences between the WFDMs and the WAFDMs as follows.

(1) The considered equations are different. A class of SFPDEs with variable coefficients and a class of time fractional diffusion equations are solved in this paper and [17], respectively.

(2) The discretizations of their fractional derivatives are different. For the WAFDMs, $\frac{\partial u}{\partial t}$ is discretized by using three-point centered difference formulas, and the time fractional derivative is discretized by using some backward difference formulas which are given by some generating functions. For the WFDMs, $\frac{\partial u}{\partial t}$ is discretized by using the explicit Euler formula, and the space fractional derivatives are discretized by using the shifted Grünwald formulas.

(3) Some similar stability results are obtained in [17] and this paper, but they are proven by using different methods.

(4) Some similar accuracy results are obtained in [17] and this paper. In this paper, the complete convergence and error estimates are given and proven. In [17], only truncating error is given.

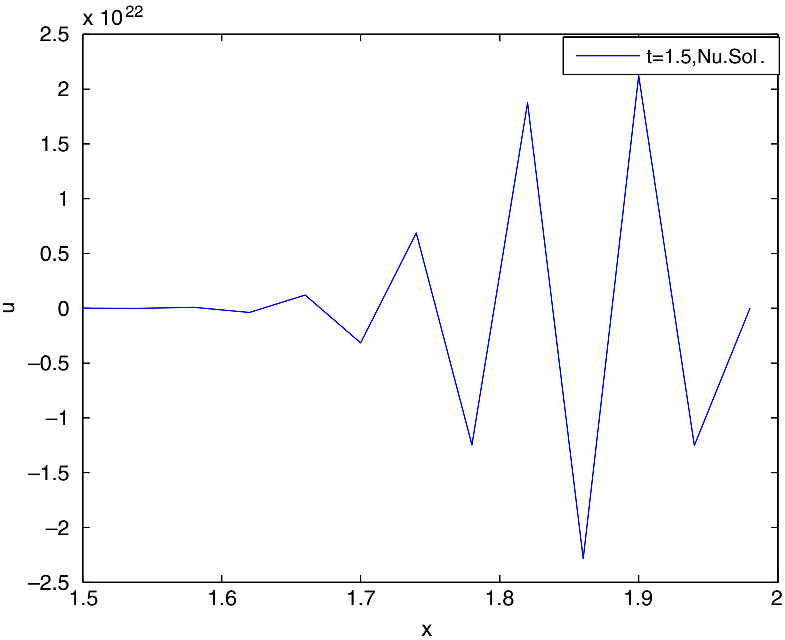

Fig. 1. The unstable numerical solution.

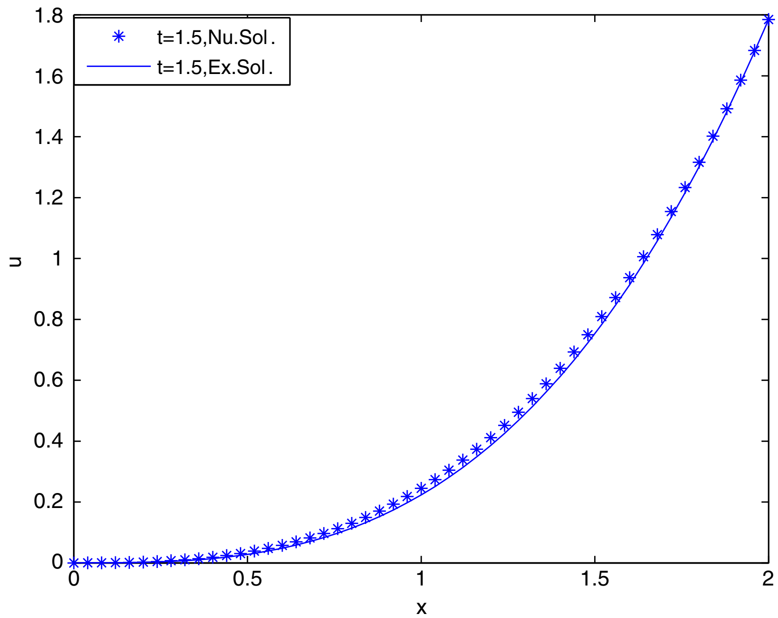

Fig. 2. Comparing the numerical solution with the exact solution ($r = 0.8$, $\tau = 0.0025$).

Remark 4.3. The considered equation (1.1) are of one dimension. But natural process is seldom one dimension. Real-world anomalous diffusion can be direction dependent, and it is not necessary to be orthogonal. In recent years, multi-dimensional fractional calculus and fractional vector calculus for fractional diffusion have been discussed in [9,18,11]. In [9], a practical alternating-direction implicit Euler method for two-dimensional fractional dispersion equations with variable coefficients was presented, first-order convergence and unconditional stability of the method were proven. Moreover, in [11], an unconditionally-stable second-order accurate finite difference method for the similar equations to those in [9] was obtained by combining the alternating-direction implicit approach with a Crank–Nicholson discretization and a Richardson extrapolation. It is a interesting problem how to extend the weighted average finite difference methods to some multi-dimensional fractional partial differential equations and vector fractional models. We will discuss it in the next step.

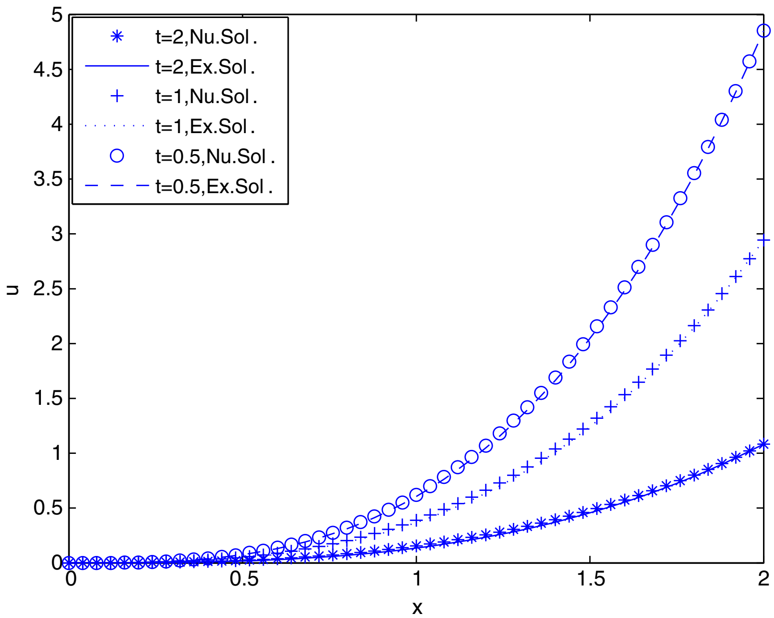

Fig. 3. Comparing the numerical solutions with the exact solutions ($r = 0.4$, $\tau = 0.025$).

5. Numerical examples

Example 5.1. Consider the initial-boundary problem of the space fractional advection–dispersion equation

$$\frac{\partial u(x,t)}{\partial t} = -v(x)\frac{\partial u(x,t)}{\partial x} + d(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + q(x,t),$$
$$u(x,0) = x^3 \quad (0 < x < 2),$$
$$u(0,t) = 0, \qquad u(2,t) = e^{-t} \quad (t > 0),$$

where the coefficients $v(x) = x/3$ and $d(x) = \Gamma(4-\alpha)x^\alpha/6$, the source term $q(x,t) = -e^{-t}x^3$. The exact solution is $u(x,t) = e^{-t}x^3$.

Choose $r = 0.8$, $\alpha = 1.8$, $\tau = 0.025$, $h = 0.04$. The condition $(\xi + \eta)\alpha + \zeta \le \frac{1}{2r-1}$ is not satisfied, therefore the WFDMs are not stable. In fact, the instability of WFDMs is shown in Fig. 1.

Choose $r = 0.8$, $\alpha = 1.8$, $\tau = 0.0025$, $h = 0.04$. The stability condition is satisfied. In Fig. 2, the resulting numerical solution is close to the exact solution.

In Fig. 3, the analytical solutions and the numerical solutions for $r = 0.4$, $\alpha = 1.8$, $\tau = 0.025$, $h = 0.04$ at $t = 2, 1, 0.5$ are shown respectively. For $r = 0.4$, the WFDMs (2.4) are unconditionally-stable. Consequently, we see that all numerical solutions are in good agreement with the analytical solutions in Fig. 3.

Example 5.2. Consider the special space fractional Fokker–Planck equation

$$\frac{\partial u(x,t)}{\partial t} = -v\frac{\partial u(x,t)}{\partial x} + \left(\frac{1}{2} + \frac{\beta}{2}\right) D\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} + \left(\frac{1}{2} - \frac{\beta}{2}\right) D\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha},$$
$$u(x,0) = \delta(x),$$
$$u(-20 + vt, t) = u(20 + vt, t) = 0,$$

where v denotes the drift of the process, that is, the mean advective velocity, D is the coefficient of dispersion, $1 < \alpha < 2$ is the order of fractional derivative, β describes the skewness of the transport process, the initial condition $\delta(x)$ is the Dirac function. This equation is a special case of the space fractional Fokker–Planck equation in [6].

We compare the WFDMs (2.4) with Method of Lines (MOL). MOL for SFPDEs is firstly proposed by Liu, Anh and Turner [6]. The scheme is as follows

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -v_i\frac{u_i(t) - u_{i-1}(t)}{h} + \frac{d_+}{h^\alpha}\sum_{k=0}^{i+1} g_k u_{i-k+1}(t) + \frac{d_-}{h^\alpha}\sum_{k=0}^{K-i+1} g_k u_{i+k-1}(t),$$

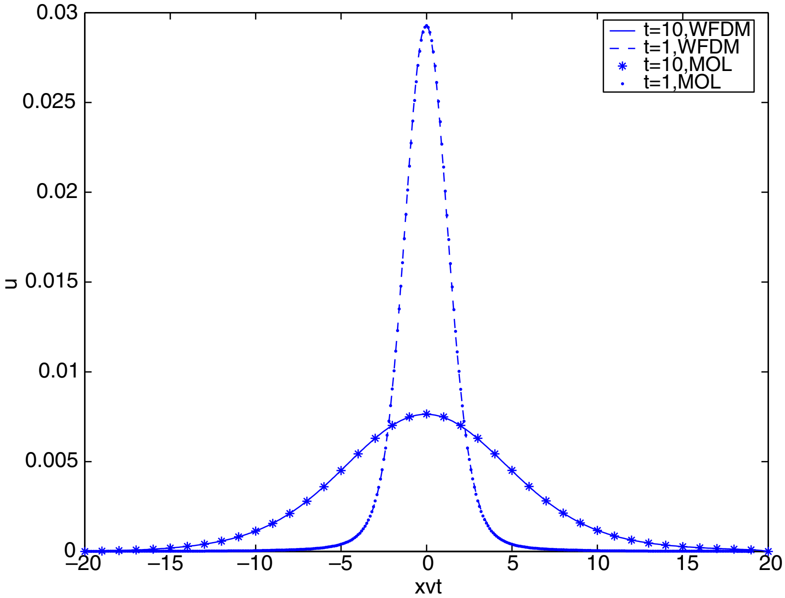

Fig. 4. Numerical comparison for Fokker–Planck equation.

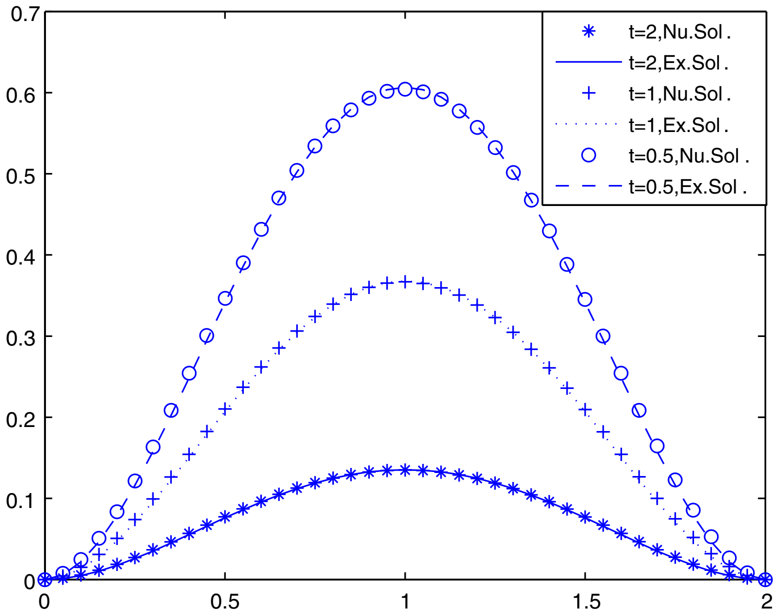

Fig. 5. Comparing the numerical solutions with the exact solutions.

where $d_+ = (\frac{1}{2} + \frac{\beta}{2})D$, $d_- = (\frac{1}{2} - \frac{\beta}{2})D$. The second-order BDF method is used to solve the above system of ordinary differential equations. The numerical solutions by using WFDMs and MOL with $v = 1$, $D = 1$, $\beta = 0$, $\alpha = 1.7$ at $t = 1, 10$ are shown in Fig. 4, here, we choose $r = 0.2$ in WFDMs. We see that the computing results obtained by WFDMs are in good agreement with those by MOL.

Example 5.3. Consider the following two-sided space fractional differential equation

$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} = -v(x)\frac{\partial u(x,t)}{\partial x} + d_+(x)\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} + d_-(x)\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} + f(x,t),\\
&u(x,0) = x^2(2-x)^2, \quad 0 < x < 2,\\
&u(0,t) = u(2,t) = 0, \quad t > 0,
\end{aligned}$$

where $v(x) = x$, $d_+(x) = \Gamma(3-\alpha)x^\alpha$, $d_-(x) = \Gamma(3-\alpha)(2-x)^\alpha$,

$$f(x,t) = -\mathrm{e}^{-t}\left[8(2-x)^2 - 24\frac{\Gamma(3-\alpha)}{\Gamma(4-\alpha)}(x^3+(2-x)^3) + 24\frac{\Gamma(3-\alpha)}{\Gamma(5-\alpha)}\right.$$
$$\left.(x^4+(2-x)^4) + 12x^3 - 4x^4\right] - \mathrm{e}^{-t}x^2(2-x)^2.$$

In Fig. 5, we show the coherence between the analytical solutions and the numerical solutions by taking $r = 0.3$, $\alpha = 1.7$, $\tau = 0.025$, $h = 0.05$ at $t = 0.5, 1, 2$.

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